LOGARITHMS

Logarithms were developed for making complicated calculations simple. However, with the advent of computers and hand calculators, doing calculations with the use of logarithms is no longer necessary. But still, logarithmic and exponential equations and functions are very common in mathematics.

Logarithms

To learn the concept of *logarithm*, consider the equality $2^3 = 8$, another way of writing this is $\log_2 8 = 3$.

It is read as "*logarithm* (abbreviated 'log') of 8 to the *base* 2 is equal to 3". Thus, $2^3 = 8$ is equivalent to $\log_2 8 = 3$. In general, we have:

Definition. *If a is any positive real number (except 1), n is any rational number and $a^n = b$, then n is called* ***logarithm*** *of b to the base a. It is written as $\log_a b$ (read as log of b to the base a). Thus,*

$a^n = b$ if and only if $\log_a b = n$.

$a^n = b$ is called the ***exponential form*** and $\log_a b = n$ is called the ***logarithmic form.***

For example:

(*i*)	$3^2 = 9,$	$\therefore \log_3 9 = 2.$
(*ii*)	$5^4 = 625,$	$\therefore \log_5 625 = 4.$
(*iii*)	$7^0 = 1,$	$\therefore \log_7 1 = 0.$
(*iv*)	$2^{-3} = \frac{1}{2^3} = \frac{1}{8},$	$\therefore \log_2 \frac{1}{8} = -3.$
(*v*)	$(10)^{-2} = \frac{1}{100} = 0.01,$	$\therefore \log_{10}(0.01) = -2.$

Remarks

- Since a is any positive real number (except 1), a^n is always a positive real number for every rational number n *i.e.* b is always a positive real number, therefore, logarithm of only positive real numbers are defined.
- Since $a^0 = 1$, $\log_a 1 = 0$ and $a^1 = a$, $\log_a a = 1$.

  Thus, remember that

  (*i*) $\log_a 1 = 0$ (*ii*) $\log_a a = 1$

  where a is any positive real number (except 1).
- If $\log_a x = \log_a y = n$ (say), then $x = a^n$ and $y = a^n$, so $x = y$.

  Thus, remember that

  $\log_a x = \log_a y \Rightarrow x = y$.
- Logarithms to the base 10 are called **common logarithms.**
- If no base is given, the base is always taken as 10.

  For example, $\log 2 = \log_{10} 2$.

Example 1. *Convert the following to logarithmic form:*

(i) $(10)^4 = 10000$ (ii) $3^{-5} = x$ (iii) $(0.3)^3 = 0.027$.

Solution. (i) $(10)^4 = 10000 \Rightarrow \log_{10} 10000 = 4$.

(ii) $3^{-5} = x \Rightarrow \log_3 x = -5$.

(iii) $(0.3)^3 = 0.027 \Rightarrow \log_{0.3}(0.027) = 3$.

Example 2. *Convert the following to exponential form:*

(i) $\log_3 81 = 4$ (ii) $\log_8 32 = \frac{5}{3}$ (iii) $\log_{10}(0.1) = -1$.

Solution. (i) $\log_3 81 = 4 \Rightarrow 3^4 = 81$.

(ii) $\log_8 32 = \frac{5}{3} \Rightarrow (8)^{5/3} = 32$.

(iii) $\log_{10}(0.1) = -1 \Rightarrow (10)^{-1} = 0.1$

Example 3. *Find the value of the following (by converting to exponential form):*

(i) $\log_2 16$ (ii) $\log_{16} 2$ (iii) $\log_3 \frac{1}{3}$

(iv) $\log_{\sqrt{2}} 8$ (v) $\log_5 (0.008)$.

Solution. (i) Let $\log_2 16 = x \Rightarrow 2^x = 16 \Rightarrow 2^x = (2)^4 \Rightarrow x = 4$,

$\therefore \quad \log_2 16 = 4$.

(ii) Let $\log_{16} 2 = x \Rightarrow 16^x = 2 \Rightarrow (2^4)^x = 2$

$\Rightarrow \quad 2^{4x} = 2^1 \Rightarrow 4x = 1 \Rightarrow x = \frac{1}{4}$,

$\therefore \quad \log_{16} 2 = \frac{1}{4}$.

(iii) Let $\log_3 \frac{1}{3} = x \Rightarrow 3^x = \frac{1}{3} \Rightarrow 3^x = (3)^{-1} \Rightarrow x = -1$,

$\therefore \quad \log_3 \frac{1}{3} = -1$.

(iv) Let $\log_{\sqrt{2}} 8 = x \Rightarrow (\sqrt{2})^x = 8 \Rightarrow (2^{1/2})^x = 2^3$

$\Rightarrow \quad 2^{x/2} = 2^3 \Rightarrow \frac{x}{2} = 3 \Rightarrow x = 6$,

$\therefore \quad \log_{\sqrt{2}} 8 = 6$.

(v) Let $\log_5(0.008) = x \Rightarrow 5^x = 0.008$

$\Rightarrow \quad 5^x = \frac{8}{1000} \Rightarrow 5^x = \frac{1}{125} \Rightarrow 5^x = (5)^{-3} \Rightarrow x = -3$,

$\therefore \quad \log_5(0.008) = -3$.

Example 4. *Find the value of x in each of the following:*

(i) $\log_2 x = 5$ (ii) $\log_4 x = 2.5$ (iii) $\log_{64} x = \frac{2}{3}$ (iv) $\log_{\sqrt{3}} x = 4$.

Solution. (i) $\log_2 x = 5 \Rightarrow x = 2^5 \Rightarrow x = 32$.

(ii) $\log_4 x = 2.5 \Rightarrow x = 4^{2.5} \Rightarrow x = (2^2)^{5/2}$

$\Rightarrow \quad x = 2^{2 \times \frac{5}{2}} \Rightarrow x = 2^5 \Rightarrow x = 32$.

(iii) $\log_{64} x = \frac{2}{3} \Rightarrow x = (64)^{2/3} \Rightarrow x = (4^3)^{2/3}$

$\Rightarrow \quad x = 4^{3\times\frac{2}{3}} \Rightarrow x = 4^2 \Rightarrow x = 16.$

(iv) $\log_{\sqrt{3}} x = 4 \Rightarrow x = (\sqrt{3})^4 \Rightarrow x = (3^{1/2})^4$

$\Rightarrow \quad x = 3^2 \Rightarrow x = 9.$

Example 5. *Solve for x:*

(i) $\log_x 243 = -5$ (ii) $\log_9 27 = 2x + 3$ (iii) $\log_2 (x^2 - 4) = 5.$

Solution. (i) Given $\log_x 243 = -5 \Rightarrow x^{-5} = 243$

$\Rightarrow \quad x^{-5} = 3^5 \Rightarrow \left(\frac{1}{x}\right)^5 = 3^5$ $\left[\because a^{-n} = \left(\frac{1}{a}\right)^n\right]$

$\Rightarrow \quad \frac{1}{x} = 3 \Rightarrow x = \frac{1}{3}.$

(ii) Given $\log_9 27 = 2x + 3 \Rightarrow 9^{2x+3} = 27 \Rightarrow (3^2)^{2x+3} = 3^3$

$\Rightarrow \quad 3^{2(2x+3)} = 3^3 \Rightarrow 2(2x + 3) = 3 \Rightarrow 4x + 6 = 3$

$\Rightarrow \quad 4x = -3 \Rightarrow x = -\frac{3}{4}.$

(iii) Given $\log_2 (x^2 - 4) = 5 \Rightarrow x^2 - 4 = 2^5$

$\Rightarrow \quad x^2 - 4 = 32 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6.$

EXERCISE 2.2

1. Convert the following to logarithmic form:

(i) $5^2 = 25$ (ii) $a^5 = 64$ (iii) $9^0 = 1$

(iv) $3^{-2} = \frac{1}{9}$ (v) $10^{-2} = 0.01$ (vi) $(81)^{3/4} = 27.$

2. Convert the following into exponential form:

(i) $\log_2 32 = 5$ (ii) $\log_3 \frac{1}{3} = -1$ (iii) $\log_8 4 = \frac{2}{3}$

(iv) $\log_{10}(0.001) = -3$ (v) $\log_2 0.25 = -2$ (vi) $\log_a \left(\frac{1}{a}\right) = -1.$

3. By converting to exponential form, find the values of:

(i) $\log_2 16$ (ii) $\log_4 8$ (iii) $\log_{10}(0.01)$

(iv) $\log_7 \frac{1}{7}$ (v) $\log_{0.5} 256$ (vi) $\log_2 0.25$

4. Solve the following equations for x:

(i) $\log_3 x = 2$ (ii) $\log_{10} x = -2$ (iii) $\log_x \frac{1}{4} = -1$

(iv) $\log_{81} x = \frac{3}{2}$ (v) $\log_4 x = -1.5$ (vi) $\log_{\sqrt{3}} (x + 1) = 2$

(vii) $\log_4 (2x + 3) = \frac{3}{2}$ (viii) $\log_2 (x^2 - 1) = 3$ (ix) $\log x = -1$

(x) $\log x = -2, 0, \frac{1}{3}.$

Answers

1. (i) $\log_5 25 = 2$ (ii) $\log_a 64 = 5$ (iii) $\log_9 1 = 0$

(iv) $\log_3 \frac{1}{9} = -2$ (v) $\log_{10} 0.01 = -2$ (vi) $\log_{81} 27 = \frac{3}{4}$

2. (i) $2^5 = 32$ (ii) $3^{-1} = \frac{1}{3}$ (iii) $(8)^{\frac{2}{3}} = 4$

(iv) $10^{-3} = 0.001$ (v) $2^{-2} = 0.25$ (vi) $a^{-1} = \frac{1}{a}$

3. (i) 4 (ii) $\frac{3}{2}$ (iii) -2 (iv) -1 (v) -8 (vi) -2

4. (i) 9 (ii) 0.01 (iii) 4 (iv) 729

(v) $\frac{1}{8}$ (vi) 2 (vii) $\frac{5}{2}$ (viii) ± 3

(ix) $\frac{1}{10}$ (x) $\frac{1}{100}, 1, \sqrt[3]{10}$

THREE STANDARD LAWS OF LOGARITHMS

- $log_a mn = log_a m + log_a n.$ $a, m, n \in R, a, m, n > 0, a \neq 1$ [Product Law]

  The above result is capable of extension *i.e.*

  $\log_a (mnp...) = \log_a m + \log_a n + \log_a p + ...$

- $log_a \frac{m}{n} = log_a m - log_a n.$ $a, m, n \in R, a, m, n > 0, a \neq 1$ [Quotient Law]

- $log_a m^n = n\, log_a m.$ $a, m, n \in R, a, m > 0, a \neq 1$ [Power Law]

Deductions

1. $\log_a a^x = x$ 2. $a^{\log_a x} = x.$

Base changing formula

$$log_a m = \frac{\log_b m}{\log_b a}. \qquad m > 0, a, b > 0, a \neq 1, b \neq 1.$$

Deductions

1. $\log_b m = \log_a m \times \log_b a.$
2. $\log_b a \times \log_a b = 1.$ (Put $m = b$ in 1)
3. $\log_b a = \frac{1}{\log_a b}.$ (Reciprocal formula)

ILLUSTRATIVE EXAMPLES

Example 1. *Expand the following:*

(i) $log_b \left(\frac{a^a b^b}{c^c d^d}\right)$ (ii) $log_b \left(\frac{4x^6}{9y^7}\right)$

Solution. (i) $\log_b \left(\frac{a^a b^b}{c^c d^d}\right) = \log_b a^a + \log_b b^b - \log_b c^c - \log_b d^d$ (Quotient law and product law)

$= a \log_b a + b \log_b b - c \log_b c - d \log_b d$ (Power law)

$= a \log_b a + b \,.\, 1 - c \log_b c - d \log_b d$ ($\because \log_b b = 1$)

$= b + a \log_b a - c \log_b c - d \log_b d.$

(ii) $\log_b \left(\frac{4x^6}{9y^7}\right) = \log_b \left(\frac{2^2 . x^6}{3^2 . y^7}\right) = \log_b 2^2 . x^6 - \log_b 3^2 . y^7$ (Quotient law)

$= \log_b 2^2 + \log_b x^6 - (\log_b 3^2 + \log_b y^7)$ (Product law)

$= 2 \log_b 2 + 6 \log_b x - 2 \log_b 3 - 7 \log_b y.$

Example 2. *Simplify the following:*

(i) $\log_{10} a + \log_{10} b^2 + \log_{10} c^3$ (ii) $\log_a a - \log_b b^2 + \log_c c^3 - \log_d d^4$

Solution. (i) $\log_{10} a + \log_{10} b^2 + \log_{10} c^3 = \log_{10} (a.b^2.c^3)$ (Product law)

(ii) $\log_a a - \log_b b^2 + \log_c c^3 - \log_d d^4$ (Power law)

$= \log_a a - 2 \log_b b + 3 \log_c c - 4 \log_d d$ (Power law)

$= 1 - 2.1 + 3.1 - 4.1 = -2.$ $(\because \log_a a = 1)$

Example 3. *Express* $\log_{10} \frac{a^2c}{\sqrt{b}}$ *in terms of* $\log_{10}a, \log_{10}b, \log_{10}c$.

Solution. $\log_{10} \frac{a^2c}{\sqrt{b}} = \log_{10} a^2c - \log_{10} \sqrt{b}$ [Quotient Law]

$= \log_{10} a^2 + \log_{10} c - \log_{10} (b)^{\frac{1}{2}}$ [Product Law]

$= 2 \log_{10} a + \log_{10} c - \frac{1}{2} \log_{10} b.$ [Power Law]

Example 4. *Evaluate:* $3 + \log_{10} (10^{-2})$.

Solution. $3 + \log_{10}(10^{-2}) = 3 + (-2) \log_{10} 10$ [Power Law]

$= 3 + (-2).1$ $[\because \log_{10} 10 = 1]$

$= 3 - 2 = 1.$

Example 5. *Evaluate the following:*

(i) $\frac{\log 125}{\log \sqrt{5}}$ (ii) $\log_6 72 - \log_6 2$ (iii) $\log_4 8 - \log_8 32.$

Solution. (i) $\frac{\log 125}{\log \sqrt{5}} = \frac{\log 5^3}{\log 5^{1/2}} = \frac{3 \log 5}{\frac{1}{2} \log 5} = 6.$

(ii) $\log_6 72 - \log_6 2 = \log_6 \frac{72}{2} = \log_6 36 = \log_6 6^2$

$= 2 \log_6 6 = 2 \times 1$ $(\because \log_a a = 1)$

$= 2.$

(iii) $\log_4 8 - \log_8 32 = \log_4 2^3 - \log_8 2^5 = \log_4 (2^2)^{3/2} - \log_8 (2^3)^{5/3}$

$= \log_4 4^{3/2} - \log_8 8^{5/3} = \frac{3}{2} \log_4 4 - \frac{5}{3} \log_8 8$

$= \frac{3}{2} \times 1 - \frac{5}{3} \times 1$ $(\because \log_a a = 1)$

$= \frac{3}{2} - \frac{5}{3} = \frac{9-10}{6} = -\frac{1}{6}.$

Example 6. *Express as a single logarithm:* $2 + \frac{1}{2} \log_{10} 9 - 2 \log_{10} 5.$

Solution. $2 + \frac{1}{2} \log_{10} 9 - 2 \log_{10} 5 = 2.1 + \frac{1}{2} \log_{10} 9 - 2 \log_{10} 5$

$= 2 \log_{10} 10 + \log_{10} (9)^{1/2} - \log_{10} (5)^2$ $[\because \log_{10} 10 = 1]$

$= \log_{10} (10)^2 + \log_{10} 3 - \log_{10} 25$

$= \log_{10} \frac{(10)^2 \times 3}{25} = \log_{10} \frac{100 \times 3}{25} = \log_{10} 12.$

Example 7. *If* $\log 7 - \log 2 + \log 16 - 2\log 3 - \log \frac{7}{45} = 1 + \log n$, *find* n.

Solution. Given $\log 7 - \log 2 + \log 16 - 2\log 3 - \log \frac{7}{45} = 1 + \log n$

$\Rightarrow \quad \log 7 - \log 2 + \log 16 - \log (3)^2 - \log \frac{7}{45} = \log 10 + \log n$ $\qquad [\because \log 10 = 1]$

$\Rightarrow \quad \log \dfrac{7 \times 16}{2 \times 3^2 \times \frac{7}{45}} = \log (10 \times n) \Rightarrow \log \dfrac{7 \times 16 \times 45}{2 \times 9 \times 7} = \log 10n$

$\Rightarrow \quad \log 40 = \log 10n \Rightarrow 40 = 10n \Rightarrow n = 4.$

Example 8. *If* $3 \log \sqrt{m} + 2 \log \sqrt[3]{n} - 1 = 0$, *find the value of* $m^9 n^4$.

Solution. Given $3 \log \sqrt{m} + 2 \log \sqrt[3]{n} - 1 = 0$

$\Rightarrow \quad \log (\sqrt{m})^3 + \log (\sqrt[3]{n})^2 = 1$

$\Rightarrow \quad \log (m^{3/2} \times n^{2/3}) = \log 10$ $\qquad [\because \log 10 = 1]$

$\Rightarrow \quad m^{3/2}.n^{2/3} = 10$, raising both sides to the power 6, we get

$(m^{3/2}.n^{2/3})^6 = 10^6$

$\Rightarrow \quad (m^{3/2})^6.(n^{2/3})^6 = 10^6 \Rightarrow m^9 n^4 = 10^6.$

Example 9. *Simplify the following:*

(*i*) $\log (\log x^2) - \log (\log x)$ $\qquad$ (*ii*) $\log_b a \,.\, \log_c b \,.\, \log_a c$ $\qquad$ (*iii*) $\log_2 (\log_2 (\log_2 16))$.

Solution. (*i*) $\log (\log x^2) - \log (\log x) = \log (2\log x) - \log (\log x) = \log \left(\dfrac{2\log x}{\log x}\right) = \log 2.$

(*ii*) $\log_b a.\log_c b.\log_a c = (\log_b a.\log_c b).\log_a c = \log_c a.\log_a c = 1.$

(*iii*) $\log_2 (\log_2 (\log_2 16)) = \log_2 (\log_2 (\log_2 2^4)) = \log_2 (\log_2 (4)) = \log_2 (\log_2 2^2) = \log_2 (2) = 1.$

Example 10. (*i*) *If* $\dfrac{\log a}{b-c} = \dfrac{\log b}{c-a} = \dfrac{\log c}{a-b}$, *prove that* $a^a \,.\, b^b \,.\, c^c = 1$.

(*ii*) *If* $\dfrac{1}{\log_a n} + \dfrac{1}{\log_c n} = \dfrac{2}{\log_b n}$, *prove that* $b^2 = ac$.

Solution. (*i*) Let $\dfrac{\log a}{b-c} = \dfrac{\log b}{c-a} = \dfrac{\log c}{a-b} = k$

$\Rightarrow \quad \log a = k(b-c);\ \log b = k(c-a);\ \log c = k(a-b)$

$\Rightarrow \quad a \log a + b \log b + c \log c = ka(b-c) + kb(c-a) + kc(a-b) = 0$

$\Rightarrow \quad \log a^a.b^b.c^c = 0 = \log 1 \Rightarrow a^a.b^b.c^c = 1.$

(*ii*) Given $\dfrac{1}{\log_a n} + \dfrac{1}{\log_c n} = \dfrac{2}{\log_b n}$

$\Rightarrow \quad \log_n a + \log_n c = 2\log_n b$ $\qquad$ (Using reciprocal formula)

$\Rightarrow \quad \log_n ac = \log_n b^2$

$\Rightarrow \quad ac = b^2$, as required.

Example 11. *If* $a = \log_x yz$, $b = \log_y zx$ *and* $c = \log_z xy$, *then prove that* $\dfrac{1}{1+a} + \dfrac{1}{1+b} + \dfrac{1}{1+c} = 1$.

Solution. $\dfrac{1}{1+a} = \dfrac{1}{1+\log_x yz} = \dfrac{1}{\log_x x + \log_x yz} = \dfrac{1}{\log_x xyz} = \log_{xyz} x.$

Similarly, $\dfrac{1}{1+b} = \log_{xyz} y$ and $\dfrac{1}{1+c} = \log_{xyz} z.$

$$\therefore \quad \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} = \log_{xyz} x + \log_{xyz} y + \log_{xyz} z$$

$$= \log_{xyz} xyz = 1.$$

Example 12. *Solve for x:*

(*i*) $\log x = \dfrac{\log 125}{\log \frac{1}{5}}$ (*ii*) $\log_2 (\log_3 x) = 4$

(*iii*) $\log_x 15\sqrt{5} = 2 - \log_x 3\sqrt{5}$ (*iv*) $\log (5x - 4) - \log (x + 1) = \log 4$

Solution. (*i*) $\log x = \dfrac{\log 125}{\log \frac{1}{5}} = \dfrac{\log(5)^3}{\log 5^{-1}} = \dfrac{3 \log 5}{(-1) \log 5} = -3$

$$\Rightarrow \quad x = 10^{-3} \Rightarrow x = \frac{1}{(10)^3} = \frac{1}{1000} = 0.001$$

(*ii*) $\log_2 (\log_3 x) = 4 \Rightarrow \log_3 x = 2^4 = 16 \Rightarrow x = 3^{16}$.

(*iii*) Given $\log_x 15\sqrt{5} = 2 - \log_x 3\sqrt{5} \Rightarrow \log_x 15\sqrt{5} + \log_x 3\sqrt{5} = 2$

$$\Rightarrow \quad \log_x (15\sqrt{5} \times 3\sqrt{5}) = 2 \Rightarrow \log_x 225 = 2$$

$$\Rightarrow \quad \log_x (15)^2 = 2 \Rightarrow 2 \log_x 15 = 2$$

$$\Rightarrow \quad \log_x 15 = 1 \Rightarrow x^1 = 15 \Rightarrow x = 15.$$

(*iv*) Given $\log (5x - 4) - \log (x + 1) = \log 4$

$$\Rightarrow \quad \log \frac{5x-4}{x+1} = \log 4$$

$$\Rightarrow \quad \frac{5x-4}{x+1} = 4 \Rightarrow 5x - 4 = 4x + 4 \Rightarrow x = 8.$$

Example 13. *Find the value of x if* $\log_{10} x - \log_{10} (2x - 1) = 1$.

Solution. Given $\log_{10} x - \log_{10} (2x - 1) = 1$

$$\Rightarrow \quad \log_{10} \frac{x}{2x-1} = 1 \Rightarrow \frac{x}{2x-1} = 10^1 \Rightarrow \frac{x}{2x-1} = 10$$

$$\Rightarrow \quad 20x - 10 = x \Rightarrow 19x = 10 \Rightarrow x = \frac{10}{19}.$$

Example 14. *Solve the following equations for x:*

(*i*) $\log_x 25 - \log_x 5 + \log_x \dfrac{1}{125} = 2$ (*ii*) $\log_x (8x - 3) - \log_x 4 = 2$

Solution. (*i*) Given $\log_x 25 - \log_x 5 + \log_x \dfrac{1}{125} = 2$

$$\Rightarrow \quad \log_x \frac{25 \times \frac{1}{125}}{5} = 2 \Rightarrow \log_x \frac{1}{25} = 2$$

$$\Rightarrow \quad \log_x \left(\frac{1}{5}\right)^2 = 2 \Rightarrow 2 \log_x \frac{1}{5} = 2$$

$$\Rightarrow \quad \log_x \frac{1}{5} = 1 \Rightarrow x^1 = \frac{1}{5} \Rightarrow x = \frac{1}{5}.$$

(*ii*) Given $\log_x (8x - 3) - \log_x 4 = 2$

$$\Rightarrow \quad \log_x \frac{8x-3}{4} = 2 \Rightarrow x^2 = \frac{8x-3}{4} \Rightarrow 4x^2 = 8x - 3$$

$\Rightarrow \quad 4x^2 - 8x + 3 = 0 \Rightarrow 4x^2 - 6x - 2x + 3 = 0$

$\Rightarrow \quad 2x(2x - 3) - 1(2x - 3) = 0 \Rightarrow (2x - 3)(2x - 1) = 0$

$\Rightarrow \quad 2x - 3 = 0, 2x - 1 = 0 \Rightarrow x = \frac{3}{2}, \frac{1}{2}.$

Example 15. *Solve for x:* $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$.

Solution. Given $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

$$\Rightarrow \quad \frac{1}{\log_x 2} + \frac{1}{\log_x 4} + \frac{1}{\log_x 16} = \frac{21}{4}$$

$$\Rightarrow \quad \frac{1}{\log_x 2} + \frac{1}{\log_x 2^2} + \frac{1}{\log_x 2^4} = \frac{21}{4} \Rightarrow \frac{1}{\log_x 2} + \frac{1}{2\log_x 2} + \frac{1}{4\log_x 2} = \frac{21}{4}$$

$$\Rightarrow \quad \frac{1}{\log_x 2}\left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{21}{4} \Rightarrow \frac{7}{4}\cdot\frac{1}{\log_x 2} = \frac{21}{4}$$

$$\Rightarrow \quad \log_x 2 = \frac{7}{4}\cdot\frac{4}{21} = \frac{1}{3}$$

$$\Rightarrow \quad x^{1/3} = 2 \Rightarrow x = 2^3 = 8.$$

EXERCISE 2.3

1. Simplify the following:

   (i) $\log a^3 - \log a^2$ (ii) $\log a^3 \div \log a^2$ (iii) $\frac{\log 4}{\log 2}$ (iv) $\frac{\log 27}{\log \sqrt{3}}$

2. Evaluate the following:

   (i) $\log (10 \div \sqrt[3]{10})$ (ii) $2 + \frac{1}{2}\log (10^{-3})$

   (iii) $2\log 10^3 + 3\log 10^{-2} - \frac{1}{3}\log 5^{-3} + \frac{1}{2}\log 4$

   (iv) $2\log 2 + \log 5 - \frac{1}{2}\log 36 - \log \frac{1}{30}$

   (v) $\log 2 + 16\log \frac{16}{15} + 12\log \frac{25}{24} + 7\log \frac{81}{80}$

3. Express each of the following as a single logarithm:

   (i) $2\log 3 - \frac{1}{2}\log 16 + \log 12$ (ii) $\frac{1}{2}\log 36 + 2\log 8 - \log 1.5$

   (iii) $7\log \frac{16}{15} + 5\log \frac{25}{24} + 3\log \frac{81}{80}$.

4. If $x = (100)^a$, $y = (10000)^b$ and $z = (10)^c$, express $\log \frac{10\sqrt{y}}{x^2 z^3}$ in terms of a, b, c.

5. If $a = \log \frac{2}{3}$, $b = \log \frac{3}{5}$ and $c = 2\log \sqrt{\frac{5}{2}}$, find the value of

   (i) $a + b + c$ (ii) 5^{a+b+c}.

6. Given $3(\log 5 - \log 3) - (\log 5 - 2\log 6) = 2 - \log n$, find n.

7. Given that $\log_{10} y + 2\log_{10} x = 2$, express y in terms of x.

8. If $a^2 = \log_{10} x$, $b^3 = \log_{10} y$ and $\frac{a^2}{2} - \frac{b^3}{3} = \log_{10} z$, express z in terms of x and y.

9. Given that $\log x = m + n$ and $\log y = m - n$, express the value of $\log\left(\frac{10x}{y^2}\right)$ in terms of m and n.

10. Solve for x:

(i) $\log x + \log 5 = 2 \log 3$ (ii) $\log_3 x - \log_3 2 = 1$

(iii) $\frac{\log 8}{\log 2} \times \frac{\log 3}{\log \sqrt{3}} = 2 \log x$.

11. Solve the following equations:

(i) $\log (2x + 3) = \log 7$ (ii) $\log (x + 1) + \log (x - 1) = \log 24$

(iii) $\log_{10}5 + \log_{10}(5x + 1) = \log_{10}(x + 5) + 1$

(iv) $\log_{10}(x + 2) + \log_{10}(x - 2) = \log_{10}3 + 3 \log_{10}4$.

12. If $\log \frac{x-y}{2} = \frac{1}{2} (\log x + \log y)$, prove that $x^2 + y^2 = 6xy$.

13. If $x^2 + y^2 = 23xy$, prove that $\log \frac{x+y}{5} = \frac{1}{2} (\log x + \log y)$.

14. Prove the following identities:

(i) $\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = 1$ (ii) $\log_b a \,.\, \log_c b \,.\, \log_d c = \log_d a$.

15. Given that $\log_a x = \frac{1}{\alpha}$, $\log_b x = \frac{1}{\beta}$, $\log_c x = \frac{1}{\gamma}$, find $\log_{abc} x$.

Hint: $\frac{1}{\alpha} = \log_a x = \frac{\log x}{\log a} \Rightarrow \log a = \alpha \log x$ etc. and

$$\log_{abc} x = \frac{\log x}{\log abc} = \frac{\log x}{\log a + \log b + \log c}.$$

16. Solve for x:

(i) $\log_3 x + \log_9 x + \log_{81} x = \frac{7}{4}$ (ii) $\log_2 x + \log_8 x + \log_{32} x = \frac{23}{15}$.

Answers

1. (i) $\log a$ (ii) $\frac{3}{2}$ (iii) 2 (iv) 6

2. (i) $\frac{2}{3}$ (ii) $\frac{1}{2}$ (iii) 1 (iv) 2 (v) 1

3. (i) log 27 (ii) log 256 (iii) log 2

4. $1 - 4a + 2b - 3c$ 5. (i) 0 (ii) 1 6. 3

7. $\frac{100}{x^2}$ 8. $\frac{\sqrt{x}}{\sqrt[3]{y}}$ 9. $1 - m + 3n$

10. (i) $\frac{9}{5}$ (ii) 6 (iii) 1000

11. (i) 2 (ii) 5 (iii) 3 (iv) 14

15. $\frac{1}{\alpha + \beta + \gamma}$ 16. (i) 3 (ii) 2